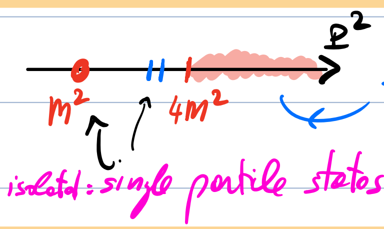
Advanced QFT @ EPFL 2024

by B. Bellazzini

This lecture is about LSZ-reduction formula, that is the connection between correlators $\langle 0|T\,\mathcal{O}(x_1)\dots\mathcal{O}(x_n)|0\rangle$ & the S-matrix elements.

Single particle states

The irreps that are isolated points in the spectrum of P^2 are the 1-particle states



the continuum of P^2 is associated to multiparticle states

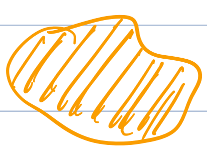
because in c.o.m $P^2 = (\sum_i E_i)^2$ $\quad E_i > m_i$

As all momentum eigenstates, they have a plane-wave e^{-ipx} overlap with suitable local fields: $\langle 0|\mathcal{O}_A^{(j_-,j_+)}(x)|p\,\sigma\rangle = z\, e^{-ipx}\, \psi_{A\sigma}(p)$

One can form finite-norm 1-particle states (true elements in Hilbert sp.) by smearing with wavepackets

(1) $$|\varphi\rangle \equiv \sum_\sigma \int \underbrace{\frac{d^3k}{(2\pi)^3}\frac{1}{2E_k}}_{d\Omega_k} \varphi_\sigma(\vec k)\,|k\,\sigma\rangle \longrightarrow \langle 0|\mathcal{O}_A(x)|\varphi\rangle = \sum_\sigma \int d\Omega_k\, \overbrace{e^{-ikx}\,\varphi_\sigma(\vec k)}^{\text{localized!}}\,\psi_{A\sigma}(k)\, z$$

This is localized as soon the Fourier transform acts on a smoothly varying function.



"1-particle blobs" = finite-region overlap with $\langle 0|\mathcal{O}_A(x)$

Examples: $F.T.(\text{const}) \propto \int dk\, e^{-ikx} 1 = 2\pi\delta(x) \to$ very local

$F.T.(\text{polyn.}) \propto \int dk\, e^{-ikx} \sum_n a_n k^n = (2\pi) \sum_n a_n i \frac{\partial^n}{\partial x^n} \delta(x) \to$ very local

$F.T.$(infinite order polynomial = analytic function) $= \int e^{-ikx} dk\, \hat{f}(k) = f(x)$

but since $k \to k + i k_I \;\forall k_I$ & $\hat{f}(k) = \int e^{ikx} f(x)\, dx \Rightarrow f(x)$ exponentially dumped

Smeared fields

conversely, we can smeared the states in $\langle 0 | O_A(x)$

scalar field example:

(2) $\int d^3x\, O(t,\vec{x})\, \chi(t,\vec{x}) \equiv O^{(\chi)}(t)$ with χ smooth solution of Klein-Gord.

$\chi(t,x) = \int \frac{d^3p}{(2\pi)^3} e^{+ipx} \hat{\chi}(\vec{p})\, 2p^0$, $p^0 = \sqrt{\vec{p}^2 + M^2}$ (i.e. χ solves $(\Box + M^2)\chi = 0$)

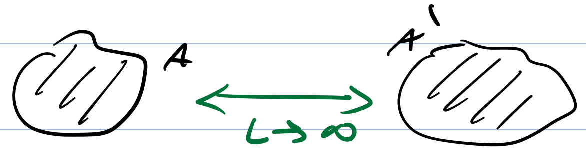
(3) $\langle 0 | O^{(\chi)}(t) | k \rangle = \int d^3x \int \frac{d^3p}{(2\pi)^3} e^{i(p-k)x} \hat{\chi}(p)\, 2p^0 z = e^{i(p^0 - k^0)t} \hat{\chi}(\vec{k})\, 2k^0 z$ time-independent if chosen $M^2 = m^2 = k^2$

$$\boxed{\langle 0 | O^{(\chi)}(t) | \varphi \rangle \underset{M=m}{=} 2 \int \frac{d^3k}{(2\pi)^3} \hat{\chi}(\vec{k})\, \varphi(\vec{k}) = \langle \chi | \varphi \rangle}$$

time-independent overlap with 1-particle blob (if setting $M=m$)

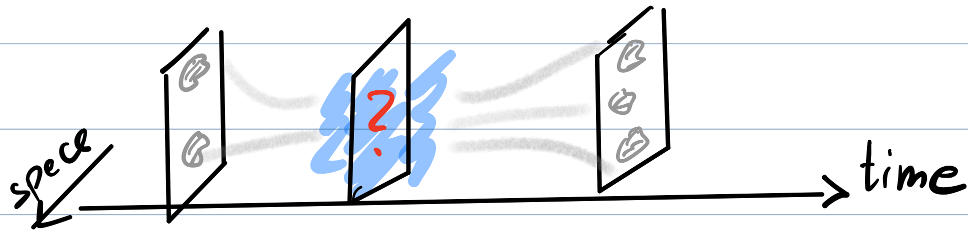
Multiparticle states & Cluster Principle

(a) if $\exists$ 1-particle blob, there should be possible to have more, separated



(b) The matrix elements factorize when $L \to \infty$ behave as independent

(c) Since we observe separated 1-particle blobs in the far past and in the far future (experimental input for certain theories, those that admit S-matrix)



We can formalize (a) + (b) + (c) assuming $\exists$ states $|\text{in}\rangle$ and states $|\text{out}\rangle$, spanning all Hilbert space $\mathcal{H} = \mathcal{H}_{in} = \mathcal{H}_{out}$

$$\boxed{\begin{aligned} \mathcal{H} = \mathcal{H}_{in} &= \text{spanned by } |\text{in}\rangle = \{ |k_1, \sigma_1, q_1; k_2 \sigma_2 q_2; \dots\rangle_{in} \} \\ &\parallel \\ \mathcal{H}_{out} &= \text{spanned by } |\text{out}\rangle = \{ |k_1, \sigma_1, q_1; k_2 \sigma_2 q_2; \dots\rangle_{out} \} \end{aligned}} \quad (4)$$

which are labelled by the same quantum numbers of the Fock space – tensor product of 1-particle states –

because all their matrix elements with operators factorize (like in the free theory would) at early / late times provided smeared with wavepackets

π: permutations

$$(5)\begin{cases} \lim\limits_{\max t_i \to -\infty} \langle 0 | T\, \mathcal{O}^{(\alpha)}(t_1) \dots \mathcal{O}^{(\alpha)}(t_n) | \varphi_1 \dots \varphi_n \rangle^{in} = \sum\limits_{\pi} \prod\limits_{i} \langle 0 | \mathcal{O}^{(\alpha)}(t_i) | \varphi_{\pi(i)} \rangle \\ \lim\limits_{\min t_i \to +\infty} {}^{out}\langle \varphi_1 \dots \varphi_n | T\, \mathcal{O}^{(\alpha)}(t_1) \dots \mathcal{O}^{(\alpha)}(t_n) | 0 \rangle = \sum\limits_{\pi} \prod\limits_{i} \langle \varphi_{\pi(i)} | \mathcal{O}^{(\alpha)}(t_i) | 0 \rangle \end{cases}$$

In other words, we assume there exists states $|in\rangle$ & $|out\rangle$ such that results of all measurements upon them return the same values of isolated 1-particle blobs (differing from free-field results only by z-factors in (5))

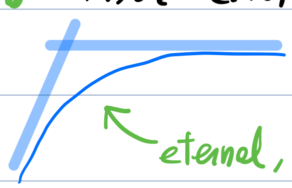
Since this is understood for any wavepacket, provided $t \to \pm\infty$, we can say

(6) $$\langle 0|\, T\, \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\,| (k_1,\sigma_1,\dots);(k_2,\sigma_2,\dots);\dots\rangle^{in} \xrightarrow{\max t_i \to -\infty} \sum_\pi \prod_{i=1}^{n} \langle 0|\mathcal{O}(x_i)|k_{\pi(i)},\sigma_{\pi(i)}\rangle$$

$${}^{out}\langle (k_1,\sigma_1,\dots);(k_2,\sigma_2,\dots);\dots\,|\,T\,\mathcal{O}(x_1)\cdots\mathcal{O}(x_n)|0\rangle \xrightarrow{\min t_i \to +\infty} \sum_\pi \prod_{i=1}^{n} \langle k_{\pi(i)},\sigma_{\pi(i)}\dots|\mathcal{O}(x_i)|0\rangle$$

Comment: • There exist in- & out-states in classical physics too

(*)



← can be labelled by initial conditions $(q,\dot{q})_{initial}$ or $(q,\dot{q})_{final}$

← eternal, time-less state (analog of Heisenberg's) is the entire trajectory

← results of asymptotic measurements asymptote those on free theory / or →

• vacuum & $|1\rangle^{in/out}$ are irreps but $|k_1\sigma_1 q_1; k_2\sigma_2 q_2,\dots\rangle^{in/out}$ are not

(the projection of $|2\text{-particle}\rangle^{in/out}$ on irreps is called partial-wave dec.)

S-matrix

(7) $${}^{out}\langle \{k_j\,\sigma_j\,q_j\}\,|\,\{k_i\,\sigma_i\,q_i\}\rangle^{in} \equiv \overset{S(\{k_i\sigma_i q_i\}\to\{k_j\sigma_j q_j\})}{\overset{\parallel}{S\text{-matrix}}} \equiv {}^{out}\langle (k_i,\sigma_i);\dots\,|\,S\,|\,(p_j\,\sigma_j)\dots\rangle^{out}$$

operator in $\mathcal{H}_{out} = \mathcal{H} = \mathcal{H}_{in}$

$$S\text{-operator} = \sum_{out} |\{k_j\,\sigma_j\,q_j\}\rangle^{out}\;{}^{out}\langle\{k_i\,\sigma_i\,q_i\}|\; S(\{k_i,\sigma_i\,q_i\} \to \{k_j\,\sigma_j\,q_j\})$$

$\Updownarrow$

$S\,|\,out\rangle = |\,in\,\rangle \longleftrightarrow |in\rangle = \sum S_{out\,in}\,|out\rangle$ by asymptotic completeness $\mathcal{H}=\mathcal{H}_{in}=\mathcal{H}_{out}$

Comments: watch-out, other definitions of $\$$-operator are common in the literature, for instance in Weinberg $\$$ acts on $\mathcal{H}_{free}$ instead

$^{free}\langle k_j\sigma_j q_j | \$ | k_i \sigma_i q_i\rangle^{free} = \$(k_i\sigma_i q_i \to k_j\sigma_j p_j)$, i.e. like sending the free blue line trajectory of (*) down)

$\left(S = \sum |free\rangle\langle free| S_{i\to j} \right)$

stability

$$(8) \begin{cases} \$|0\rangle^{out} = \underline{|0\rangle^{in} = |0\rangle^{out}} \\ \$|k,\sigma,q\rangle^{out}_{1\text{-particle}} = \underline{|\{k,\sigma,q\}\rangle^{in}_{1\text{-particle}} = |k,\sigma,q\rangle^{out}_{1\text{-particle}}} \end{cases}$$

i.e. $|in\rangle = |out\rangle$ for $|0\rangle$, $|1\rangle$

Unitarity

$$\mathcal{H} = \mathcal{H}_{in} = \mathcal{H}_{out} \Longrightarrow \mathbb{1} = \sum_\alpha |\alpha\rangle^{in\ in}\langle\alpha| = \sum_\beta |\beta\rangle^{out\ out}\langle\beta|$$

$$\alpha = \{k_i, \sigma_i, q_i; \dots\} \quad \beta = \dots$$

$\Rightarrow$ sandwich with $^{out}\langle\gamma|\mathbb{1}|\rho\rangle^{out}$ or $^{in}\langle\gamma|\mathbb{1}|\rho\rangle^{in}$ (suitable measure left understood & suitable $\delta_{\gamma\rho}$ too)

$$(8)\quad \boxed{\delta_{\gamma\rho} = \$_{\gamma\alpha}\, \$^*_{\rho\alpha} \qquad \delta_{\gamma\rho} = \$^*_{\beta\gamma}\, \$_{\beta\rho} \Longrightarrow \$\$^+ = \mathbb{1} = \$^+\$}$$

[example normalization: $d\alpha\,|\alpha\rangle\langle\alpha| \equiv |0\rangle\langle 0| + \overbrace{\sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{d^3k_i}{(2\pi)^3}\frac{1}{2\omega(k_i)}}^{d\alpha}\,\overbrace{|k_1,\dots,k_n\rangle\langle k_1,\dots k_n|}^{\alpha}\left(\frac{1}{n!}\right)$

e.g. $\delta_{\gamma\rho} = \langle k_3 k_4 | k_1 k_2\rangle = (2\pi)^3\delta^3(k_1-k_3)2\omega_1\,(2\pi)^3\delta^3(k_2-k_4)2\omega_2 + (1\leftrightarrow 2)$ in the 2-particle subspace]

— LSZ - Reduction —

$$G(\underbrace{p_1,\dots p_n}_{p_i^0>0};\underbrace{k_1,\dots k_m}_{k_i^0>0}) \equiv \prod_i\int d^4x_i\, e^{ip_i x_i}\prod_j\int d^4y_j\, e^{-ik_j y_j}\langle 0|T\,\mathcal{O}(x_1)\dots\mathcal{O}(x_n)\,\mathcal{O}(y_1)\dots\mathcal{O}(y_m)|0\rangle$$

$$(9)\quad \xrightarrow[\substack{p_i^2\to m_i^2 \\ k_j^2\to m_j^2}]{} \prod_{i,j}\frac{iZ_i}{p_i^2-m_i^2+i\epsilon}\,\frac{iZ_j}{k_j^2-m_j^2+i\epsilon}\,\underline{\$(k_i\sigma_i q_i \to p_j\sigma_j q_j)} + \text{subleadings}$$

residue of simple poles in F.T. correlators when going onshell

(10) $$\boxed{S\text{-matrix} = \lim_{\substack{p_i^2 \to m_i^2 \\ k_j^2 \to m_j^2}} \prod_{i,j} \frac{(p_i^2 - m_i^2 + i\epsilon)}{i Z_i} \frac{(k_j^2 - m_j^2 + i\epsilon)}{i Z_j} G(p_i, k_j)}$$

since F.T. depends on 4-momentum, the on-shell limit is seen as $p^0 \to \sqrt{\vec{p}^2 + m_i^2}$

Proof: we are after singularities of F.T. of $\langle 0 | T \mathcal{O}(x_1) \dots \mathcal{O}(x_n) | 0 \rangle$; they arise either at

- $x_i \to x_j$ regions of integration: UV-singularities, $(p_i^2 \to \infty)$ not what we're after
- from $(x_i \to \infty)$-region of integration: IR-singularities associated particles reaching inf.

Example:

- $\int_{-L}^{L} dx \, e^{+ikx} \, 1 = \frac{-i}{k} \left(e^{ikL} - e^{-ikL} \right) = \frac{2L}{k} \sin kL$ finite, it diverges as $L \to \infty$

The $\int_{-\infty}^{+\infty} dx \, e^{ikx} \, 1 = 2\pi \delta(k)$ — very singular — obtained by $\int_{L}^{+\infty}$ & $\int_{-\infty}^{-L}$ regions of int.

namely $\int_{L}^{\infty} e^{ikx} dx = \frac{+i e^{ikL}}{k + i\epsilon}$ $\int_{-\infty}^{-L} dx \, e^{ikx} = \frac{-i e^{-ikL}}{k - i\epsilon}$ that added to above give $2\pi\delta(k)$.

Let's thus focus on the asymptotic integration regions: $\boxed{x_i^0 \to +\infty \quad y_i^0 \to -\infty}$

(11) $$\langle 0 | T \mathcal{O}(x_1) \dots \mathcal{O}(x_n) \mathcal{O}(y_1) \dots \mathcal{O}(y_m) | 0 \rangle \underset{\substack{x_i^0 \to +\infty \\ y_i^0 \to -\infty}}{=} \langle 0 | T(\mathcal{O}(x_1) \dots \mathcal{O}(x_n)) \, T(\mathcal{O}(y_1) \dots \mathcal{O}(y_m)) | 0 \rangle$$

insert resolution of identity here

(12) $$\mathbb{1} = \mathbb{1} \, \mathbb{1} = \left(\sum_\alpha | \alpha \rangle^{out\, out} \langle \alpha | \right) \left(\sum_\beta | \beta \rangle^{in\, in} \langle \beta | \right) = \sum_{\alpha \beta} | \alpha \rangle^{out\, out} \langle \alpha | \beta \rangle^{in\, in} \langle \beta |$$

(13) $$= \int \underbrace{\prod_i \frac{d\Omega_i}{n!} \dots \prod_j \frac{d\Omega_{w_j}}{m!}}_{dLIPS} | q_1 \dots q_n \rangle^{out} \cdot {}^{out}\langle q_1 \dots q_n | w_1 \dots w_m \rangle^{in} \cdot {}^{in}\langle w_1 \dots w_m |$$

$S(w_1 \dots w_n \to q_1 \dots q_n) = S(w \to q)$

(the other labels suppressed for simplicity)

$$(14)\quad \boxed{\langle 0|T\,\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\,\mathcal{O}(y_1)\ldots\mathcal{O}(y_m)|0\rangle\Big|_{\substack{x_i^0\to+\infty\\ y_i^0\to-\infty}} = \int d\Omega_{q_i}\, d\Omega_{w_j}\, \langle 0|T(\mathcal{O}(x_1)\ldots\mathcal{O}(x_n))|q_1\ldots q_n\rangle^{out}\, S(w\to q)\,. \ {}^{in}\langle w_1,\ldots w_m|T(\mathcal{O}(y_1)\ldots\mathcal{O}(y_m))|0\rangle}$$

In this region we can use the factorization property (6):

For scalars first

$$(15)\quad \begin{cases} \langle 0|T(\mathcal{O}(x_1)\ldots\mathcal{O}(x_n))|q_1\ldots q_n\rangle^{out} \xrightarrow[x_i^0|_{min}\to+\infty]{} \sum_\pi \prod_i \langle 0|\mathcal{O}(x_i)|q_{\pi(i)}\rangle = \sum_\pi \prod_i z\, e^{-iq_{\pi(i)}x_i} \\ {}^{in}\langle w_1,\ldots w_m|T(\mathcal{O}(y_1)\ldots\mathcal{O}(y_m))|0\rangle \xrightarrow[y_i^0|_{max}\to-\infty]{} \sum_\pi \prod_i z\, e^{+iw_{\pi_i}y_i} \end{cases}$$

← for scalars first

each of these terms, plugged into the Fourier transformed, give:

$$(16)\quad \int_{T_{min}}^{\infty} dt_i \int d^3x_i \int d^3\Omega_{q_{\pi(i)}}\, e^{ip_ix_i}\, e^{-iq_{\pi(i)}x_i}\, z = \int_{T_{min}}^{\infty} dt_i \underbrace{\int d^3\Omega_{q_{\pi(i)}}\, (2\pi)^3\delta^3(\vec p_i - \vec q_{\pi(i)})}_{\frac{1}{2E(\vec p_i)}\,\equiv\, E_i = \sqrt{\vec p_i^{\,2}+m^2}}\, e^{i(p_i^0-q^0_{\pi(i)})t_i}\, z$$

T_{min} ← time where factorization holds → T_{min}

$$(17)\quad = \int_{T_{min}}^{\infty} dt_i\, \frac{1}{2E_i}\, e^{i(p_i^0-E_i)t_i}\, z$$

⇐ Oscillating integrals → average to 0 unless $p_i^0 \to E_i$ (on-shell condition) where they diverge because in resonance!

$$(18)\quad \underset{\epsilon\to0^+}{=} \int_{T_{min}}^{\infty} dt_i\, \frac{1}{2E_i}\, e^{i(p^0-E_i+i\epsilon)t_i}\, z = \frac{i\, e^{i(p^0-E_i)T_{min}}}{2E_i(p_i^0-E_i+i\epsilon)}$$

with $\epsilon\to0^+$ at the end

(not needed if smearing in $\vec p$)

We are interested in the singularity as $p_i^0 \to E_i = \sqrt{\vec p^{\,2}+m^2} \Rightarrow \frac{1}{2E_i} = \frac{1}{p_i^0+E_i} + O(p_i^0-E_i)$

Likewise, the $\exp(i(p^0-E_i)T_{min}) = 1 + O(p^0-E_i)$.

Since we want to extract the singular part, we get

(19) $$\boxed{\dots = \frac{i\, z}{(p_i^0+E_i)(p_i^0-E_i+i\varepsilon)} + \text{regular term} = \frac{i\, z}{p_i^{0\,2}-E_i^2+i\varepsilon} + \text{regular terms}}$$

← singular part independent of T_{min}

Analogously, the same happens for the $(-\infty, T_{max})$-integration region, using (15) for the overlap with in-states $\dots |\omega_i\rangle^{in}\ {}^{in}\langle\omega_i| T\,\mathcal{O}(y_1)\dots\mathcal{O}(y_m)|0\rangle$

(20) $$\int_{-\infty}^{T_{max}} dt_j \int d^3y_j \int d^3Q_{\omega_{\pi(j)}}\, e^{-iK_j y_j}\, e^{i\omega_{\pi(j)} y_j}\, z \underset{\varepsilon\to 0^+}{=} \int_{-\infty}^{T_{max}} dt_j\, e^{-i(K_j^0-E_j+i\varepsilon)t_j} \frac{z}{2E_j}$$

(21) $$= +i\, \frac{e^{-i(K_j^0-E_j)T_{max}}\, z}{2E_j(K_j^0-E_j+i\varepsilon)} \boxed{= \frac{i\, z}{K_j^{0\,2}-E_j^2+i\varepsilon} \quad + \text{regular terms in } K_j^0\to E_j}$$

Both asymptotic regions generate simple poles, $K_j^{0\,2}-E_j^2 = K_j^{0\,2}-\vec{K}_j^2-m^2 = K^2-m^2$

(22) $$\longrightarrow \boxed{\frac{i\, z}{K^2-m^2+i\varepsilon}} + \text{regular terms}$$

Putting everything together:

the external frequency K_j^0 & p_i^0 are tuned to the on-shell value exactly to resonate with the frequency E_i of 1-particle states when $t\to\mp\infty$.
Everything else is averaged to zero (or smeared out by wavepackets).
Each term gives a simple pole, there are $n!$ & $m!$ permutations to consider but they gives exactly the same contributions (the momenta are dummy integration variables) that cancel against the $1/n!$ & $1/m!$ factor from phase-space volume of identical particles in Eq. (13).

In summary, LSZ-formula is

$$\int \prod d^4x_i \int \prod d^4y_j\, e^{+ip_i\cdot x_i}\, e^{-ik_j\cdot x_j} \langle 0|T\,\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\,\mathcal{O}(y_1)\ldots\mathcal{O}(y_m)|0\rangle \xrightarrow[\text{on-shell } i,j]{} \prod \left(\frac{iZ}{p_i^2-m^2+i\varepsilon}\right)\left(\frac{iZ}{k_j^2-m^2+i\varepsilon}\right) S^{\text{connected}}(k_j \to p_i)$$

$(p_i^0, k_j^0 > 0)$

or equivalently

(23)
$$\prod_{j=1}^{n+m}\left[\frac{-i(k_j^2-m^2+i\varepsilon)}{Z}\right] F.T.\,\langle 0|T\,\mathcal{O}_1\ldots\mathcal{O}_{n+m}|0\rangle \xrightarrow[\text{on-shell}]{} S^{\text{connected}}(k_j \to -k_i)$$

k_j: those w/. $k_j^0>0$ & on-shell; $-k_i$: those with $k_i^0<0$ & on-shell

$\left(F.T. = \int dx\, e^{-ikx}\right)$

$$\int \prod_{i=1}^{n+m} d^4x_i\, e^{-k_i\cdot x_i}\, -i\left(-\Box_{x_i}-m^2+i\varepsilon\right) \langle 0|T\,\mathcal{O}(x_1)\ldots\mathcal{O}(x_{n+m})|0\rangle \longrightarrow S^{\text{connected}}(k_j \to -k_i)$$

Some comments are in order:

(a) We have found some of the singularities, those due to 1-particle overlaps $\prod \langle 0|\mathcal{O}|\text{1-particle}\rangle$. What about the case of non-isolated points in P^2? E.g. $\langle 0|\mathcal{O}|\text{2-particle}\rangle^{\text{out}}\, {}^{\text{out}}\langle \text{2-particle}|\ldots$? Non-isolated m^2 in P^2-spectrum would give analogous contributions $\int dm^2 \frac{Z(m^2)}{p^2-m^2+i\varepsilon}\langle p|\ldots$ but integrated over continuum of m^2

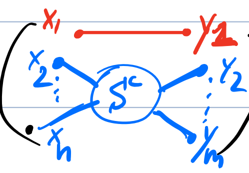
↳ smeared out singularity, e.g. pole → $\log(p^2-m^2)$ in the region $m^2 \simeq p^2$ where $\rho(m^2) \simeq$ const. ⟹ Weaker singularities than poles

(b) what about more fields than particles?

This comes about when inserting $\mathbb{1} = \sum_{\alpha\beta} |\alpha\rangle^{out\ out}\langle\alpha|\beta\rangle^{in\ in}\langle\beta|$ and selecting $|\alpha\rangle^{in/out}$ with fewer partic.
That would have produced terms with $\langle 0|T\mathcal{O}(x_1)\mathcal{O}(y_1)|0\rangle$ factored out (which decays away when $x_1 - x_2$ are separated). The F.T. produce $(2\pi)^4\delta^4(k+p)\int \frac{i}{k^2-m^2+i\epsilon}\rho(m^2)\,dm^2 \supset (2\pi)^4\delta^4(k+p)\, i\,\frac{Z(\bar m^2)}{k^2-\bar m^2+i\epsilon}$ so that the action of $(k^2-\bar m^2)$ & $(p^2-\bar m^2)$ kills this term as $k^2\to\infty$ & $p^2\to m^2$.

The LSZ - projects on the connected part of S-matrix

(24) $$(k_1^2-m^2)(p_1^2-m^2)\prod_{i,j}^{n-1,m-1}(p_i^2-m^2)(k_j^2-m^2)\Big(\ \cdots\ \Big) \xrightarrow[\text{on-shell}]{} 0$$



(likewise, if more particles than fields $\to 0$ because $\langle 0|1\text{-particle}\rangle = 0$

(we are also assuming $\langle\mathcal{O}\rangle = 0$, as requested by cluster dec.)

(c) what if we flip the sign of p_i^0 or k_j^0? say $p^0 \to -p^0$

we would have gotten oscillating integrals of the type

(25) $$\int_{T_{min}}^{\infty} dt\ e^{-i\underbrace{(p^0+q^0)}_{\text{both positive, never zero}}t} \longrightarrow \text{doesn't produce a pole.}$$

However, they would have produce a particle in the ingoing state

(26) $$\int_{-\infty}^{T_{max}} dt\ e^{-i(p^0-\omega^0)t}$$

Example of crossing symm.

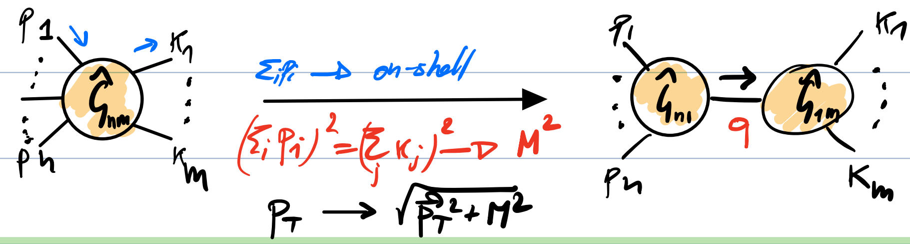
The sign of p^0 determines whether we extract an in-going or outgoing state out of the Green's functions.
Moreover, by CPT, it must be the antiparticle because we are using the same operatore $\mathcal{O}$:

$$\langle 0|\,\mathcal{O}(0)\,|p\,q\rangle = z \underset{\text{antiunitary}}{=} \langle 0|\,\underbrace{U_{CPT}\,\mathcal{O}\,U^{-1}_{CPT}}_{\mathcal{O}^\dagger}\,\underbrace{U_{CPT}|p\,q\rangle}_{|p\,\bar{q}\rangle\times\text{phase}}^{*} = \langle p\,\bar{q}\,|\,\mathcal{O}\,|\,0\rangle\cdot\text{phase}$$

antiparticle (if spin, $\sigma \to -\sigma$)

— Factorization —

We have discussed singularities associated to a $p_i^2 \to m_i^2$ where p_i is momentum of the field $\mathcal{O}(x)$. A similar story actually repeats when combinations of p_i hit a single pole certain particle mass: $(\sum_i p_i)^2 \to M^2$. In this case the correlator factorizes

(27) 

$\begin{pmatrix} K_T = \sum_i K_i \\ P_T = \sum_i p_i \end{pmatrix}$

($\hat{G}$: F.T. with overall $(2\pi)^4\delta^4(\sum p_i)$ removed)

[Reminder: invariance under translations imply that F.T. $\langle 0|T\,\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)|0\rangle$
$\propto (2\pi)^4\delta^4(\sum_i^M p_i)\,\hat{G}(p_1\ldots p_M)$: $\int \prod_{i=1}^M dx_i e^{ip_ix_i}\langle 0|T\,\mathcal{O}(x_1)\ldots\mathcal{O}(x_M)|0\rangle$

$$= \int \prod_{i=1}^M dx_i e^{ip_ix_i}\langle 0|T\,\mathcal{O}(x_1-x_M)\,\mathcal{O}(x_2-x_M)\ldots\mathcal{O}(0)\,|0\rangle = \qquad (x_i - x_M = \bar{x}_i \;\; i\neq M \quad x_M = \bar{x}_M)$$

$$= \int\!\!\int d\bar{x}_M\, e^{i(p_1+\ldots+p_M)x_M}\prod_{i=1}^{M-1} d^4\bar{x}_i\, e^{ip_i\bar{x}_i}\langle 0|T\,\mathcal{O}(\bar{x}_1)\,\mathcal{O}(\bar{x}_2)\ldots\mathcal{O}(\bar{x}_{M-1})\,\mathcal{O}(0)\,|0\rangle$$

$$= (2\pi)^4\delta^4(\sum p_i)\,\hat{G}(p_i) \quad \text{where } \hat{G} \text{ is F.T. of remaining } M-1 \text{ variables}$$]

The proof is basically in the same spirit as above: among the integration regions in the F.T. focus on $\min x_i^0 > \max y_i^0$ (e.g. $x_1^0 > x_2^0 > \ldots > x_n^0 > y_1^0 > \ldots > y_m^0$ + permutations) and insert on a complete set of state among which we single out the one-particle contribution $\mathbb{1} = \sum \mathbb{1}_{1\text{-particle}} + \sum \mathbb{1}_{2\text{-partic.}} + \ldots$; $\mathbb{1}_{1\text{-partic.}} = \int d^3 q\, |q\rangle\langle q|$ (for 1-particle $|q\rangle^{in} = |q\rangle^{out}$ do not need to distinguish:

$$(28)\quad \underbrace{\langle 0| T\big(\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\, \mathcal{O}(y_1)\ldots\mathcal{O}(y_m)\big)|0\rangle}_{\min x_i^0 > \max y_j^0} = \langle 0| T\big(\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)\big) \underbrace{\;}_{\int d^3 q\, |q\rangle\langle q| + \text{ more particles}} T\big(\mathcal{O}(y_1)\ldots\mathcal{O}(y_m)\big)|0\rangle$$

$$(29)\quad \begin{cases} \langle 0|T\,\mathcal{O}(x_1)\ldots\mathcal{O}(x_n)|q\rangle \overset{\text{translations}}{=} \langle 0|T\,\mathcal{O}(x_1-x_n)\ldots\mathcal{O}(x_{n-1}-x_n)\,\mathcal{O}(0)|q\rangle\, e^{-iqx_n} \\ \langle q|T\,\mathcal{O}(y_1)\ldots\mathcal{O}(y_m)|0\rangle = \langle q|T\,\mathcal{O}(0)\,\mathcal{O}(y_2-y_1)\ldots\mathcal{O}(y_m-y_1)|0\rangle\, e^{iqy_1} \end{cases}$$

$$(30)\quad \boxed{\begin{aligned}&\int \prod_{i=1}^{n} d^4x_i \prod_{j=1}^{m} d^4y_j\, \theta(\min x_i^0 - \max y_j^0)\, e^{ip_i x_i - ik_j y_j} \int d^3 q\, e^{-iqx_n} e^{iqy_1}\,\cdot \\ &\cdot \langle 0|T\,\mathcal{O}(x_1-x_n)\ldots\mathcal{O}(x_{n-1}-x_n)\mathcal{O}(0)|q\rangle \langle q|T\,\mathcal{O}(0)\,\mathcal{O}(y_2-y_1)\ldots|0\rangle\end{aligned}}$$

so that changing variables $\underline{x_i - x_n = \bar{x}_i}$ $\underline{y_j - y_1 = \bar{y}_j}$ $\begin{pmatrix} i \neq n \\ j \neq 1 \end{pmatrix}$

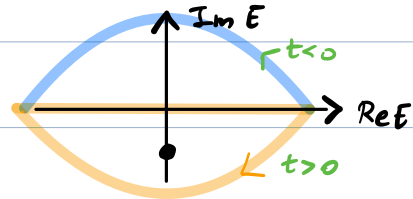
$\Big(\min x_i^0 = x_n^0 + \min(\underbrace{x_i^0 - x_n^0}_{\bar{x}_i^0})$; $\max y_j^0 = y_1^0 + \max(\underbrace{y_j^0 - y_1^0}_{\bar{y}_j^0})$; (e.g. $x_1 < x_2 < x_n < \ldots \le x_2$, $x_1 - x_n < x_2 - x_n < \ldots \le x_2 - x_n$, $\min x_i = x_2 - x_n$, $\min(x_i - x_n) = \min x_i - x_n$) $\Big)$

$$(31)\quad \int \prod_{i=1}^{n-1} d^4\bar{x}_i\, e^{ip_i\bar{x}_i} \prod_{j=2}^{m} d^4\bar{y}_j\, e^{-ik_j\bar{y}_j} \int d^4x_n\, d^4y_1\, \theta\big(x_n^0 - y_1^0 + \min(\bar{x}_i^0) - \max(\bar{y}_j^0)\big)$$

$$\cdot \int d^3 q\, e^{i\big(\underbrace{\sum_{i=1}^{n}(p_i)}_{P_T} - q\big)x_n}\, e^{-i\big(\underbrace{\sum_{j=1}^{m}(k_j)}_{K_T^j} - q\big)y_1} \langle 0|T\,\mathcal{O}(\bar{x}_1)\ldots\mathcal{O}(\bar{x}_{n-1})\,\mathcal{O}(0)|q\rangle\langle q|T\,\mathcal{O}(0)\,\mathcal{O}(\bar{y}_2)\ldots|0\rangle$$

and it's convenient to use an integral expression for the step funct.

(32) $$\boxed{\theta(t) = \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, \frac{i}{E+i\epsilon}\, e^{-iEt}}$$



(33) $$\Rightarrow (31) = \int d^3\Omega_q \int_{-\infty}^{+\infty} \frac{dE}{2\pi}\, \frac{i}{E+i\varepsilon} \int d^4x_n\, d^4y_1 \Big(e^{i(p_T^0 - q^0 - \varepsilon)x_n^0}\, e^{-i(k_T^0 - q^0 - \varepsilon)y_1^0} .$$

$$\cdot\, e^{i(\vec{p}_T - \vec{q})\vec{x}_n}\, e^{-i(\vec{k}_T - \vec{q})\vec{y}_1} \Big) \int \prod_{i=1}^{n-1} d^4\bar{x}_i\, e^{ip_i\bar{x}_i} \prod_{j=2}^{m} d^4\bar{y}_j\, e^{-ik_j\bar{y}_j}\, e^{-iE(m_{in}-m_{ax})} \langle 0|T\phi(\bar{x}_1)\ldots\phi(\bar{x}_{n-1})\phi(0)|q\rangle\langle q|T\ldots\rangle$$

such that integration over x_n & y_1 is trivial

int. x_n & y_1 $\to$ $$\boxed{(2\pi)^3\delta^3(\vec{p}_T - \vec{q})\,(2\pi)^3\delta^3(\vec{k}_T - \vec{q})\,(2\pi)\delta(p_T^0 - q^0 - E)\,(2\pi)\delta(k_T^0 - q^0 - E)}$$

The integration over $d^3\vec{q}$ & dE is now simple too $\boxed{\vec{q} = \vec{p}_T}$

(34) $$\boxed{\vec{p}_T = \vec{k}_T \;\left(= \Sigma_i \vec{p}_i = \Sigma_j \vec{k}_j\right) \quad \text{and} \quad E = p_T^0 - q^0 = k_T^0 - q_T}$$

(which imply energy-mom. conservation) (recall that $q^0 = \sqrt{\vec{q}^{\,2} + M^2}$) $\quad \vec{q}^{\,2} = \vec{p}_T^{\,2}$

(35) $$\int \prod_{i=1}^{n-1} d^4\bar{x}_i\, e^{ip_i\bar{x}_i} \prod_{j=2}^{m} d^4\bar{y}_j\, e^{-ik_j\bar{y}_j}\, \frac{i\, e^{-i(p_T^0 - q^0)(m_{in}-m_{ax})}}{(p_T^0 - q^0 + i\varepsilon)\,2q^0}\, (2\pi)^4\delta^4(p_T - k_T).$$

$$q^0 = \sqrt{\vec{q}^{\,2} + M^2} = \sqrt{\vec{p}_T^{\,2} + M^2}$$

$$\cdot\, \langle 0|T\phi(\bar{x}_1)\ldots\phi(\bar{x}_{n-1})\phi(0)|q\rangle\langle q|\phi(0)\,\phi(\bar{y}_2)\ldots\phi(\bar{y}_m)|0\rangle$$

Since we are interested in the leading singularity as $p_T^0 \to \sqrt{\vec{p}_T^{\,2}+M^2}$

(36) $$\int \prod_{i=1}^{n-1} d^4\bar{x}_i\, e^{ip_i\bar{x}_i} \prod_{j=2}^{m} d^4\bar{y}_j\, e^{-ik_j\bar{y}_j}\, \frac{i\,(2\pi)^4\delta^4(p_T-k_T)}{(p_T^2-M^2+i\varepsilon)}\left(1+\text{less-singular as } p_T^0\to\sqrt{\vec{p}_T^{\,2}+M^2}\right)\cdot$$

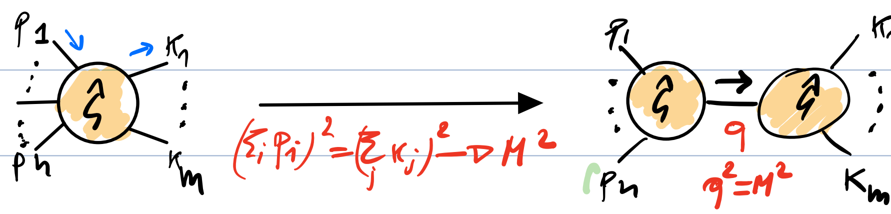
$$\cdot\langle 0|T\,\mathcal{O}(\bar{x}_1)\dots\mathcal{O}(\bar{x}_{n-1})\,\mathcal{O}(0)|q\rangle\langle q|\mathcal{O}(0)\,\mathcal{O}(\bar{y}_2)\dots\mathcal{O}(\bar{y}_m)|0\rangle$$

that is, pulling out the momentum conservation $(2\pi)^4\delta^4(\Sigma p_i-\Sigma k_i)$

FACTORIZATION

(37) $$\hat{G}(p_1,\dots,p_n;-k_1,\dots,-k_m)\xrightarrow[(\sum_i p_i)^2\to M^2]{}\hat{G}(p_1,\dots,p_n;q)\,\frac{i}{(\Sigma p_i)^2-M^2+i\varepsilon}\,\hat{G}(-k_1,\dots,-k_m;-q)$$

$q^2=M^2$ mass particle $|q\rangle$



Remark:

This is a statement of factorization of field correlators, but we can translate it to factorization of scattering amplitude by further taking residues of external legs $k_i^2\to m_i^2$ $p_i^2\to m_i^2$.
(The only caveat is for certain kinematics (e.g. M large) all these conditions can't be satisfied by all real momenta and need some complex kinematics, a statement in complex plane in general).

LSZ: General Spin

Let's consider a field $\mathcal{O}_A^{(j_-,j_+)}(x)$ that has non-vanishing overlap with spin-$\mathfrak{s}$ particles (and anti-particles)

clebsch-Gord.

$$(38)\quad \begin{cases} \langle 0|\,\mathcal{O}_A^{(j_-,j_+)}(x)\,|p\,\sigma\rangle_s = z^{(s)}(m^2)\, e^{-ipx}\, \psi_A^{(s)\sigma}(p) & \left(\text{where } \psi_A^{(s)\sigma}(p) = D_{AB}^{(j_-,j_+)}(\Lambda_p)\, c_B^{(s)\sigma}\right) \\ {}_{s}\langle p\,\sigma|\,\mathcal{O}_A^{(j_-,j_+)}(x)\,|0\rangle = z^{(s)}(m^2)\, e^{ipx}\, \psi_A^{(s)-\sigma}(p)\ (\cdot\text{phase}) & \end{cases}$$

antipart.

$((j_-,j_+)$ often left understood$)$

$$\left[\!\left[\ U_{CPT}^{-1}\,\mathcal{O}_A(x)\,U_{CPT} = \underbrace{\eta_{CPT}}_{\text{phase}}\,\mathcal{O}_A^{+}(-x) \ \Rightarrow\ {}^{\text{anti}}\langle p\,\sigma|\,\mathcal{O}_A(x)|0\rangle = \langle 0|\,\mathcal{O}_A^{+}(x)\,|p\,\sigma\rangle_{\text{anti}}^{*}\right.\right.$$

$$\left.\left.= \eta_{CPT}\langle 0|\,U_{CPT}^{-1}\mathcal{O}_A(x)\,U_{CPT}|p\,\sigma\rangle_{\text{anti}}^{*} \underset{\substack{CPT\\ \text{antiunitary}}}{=} (\eta_{CPT}\cdot \underbrace{\eta^{\sigma}}_{\text{phase}})\langle 0|\,\mathcal{O}_A(-x)\,|p\,{-\sigma}\rangle_{\text{particle}} = \underbrace{(\eta_{CPT}\cdot\eta^{\sigma})}_{\text{phase}^{(\sigma)}}\, e^{ipx}\,\psi_A^{s-\sigma}(p)\ \right]\!\right]$$

the main difference is in Eq. (45), that now becomes

$$(39)\quad \boxed{\langle 0|\,T(\mathcal{O}_{A_1}(x_1)\dots\mathcal{O}_{A_n}(x_n))\,|q_1\sigma_1\dots q_n\sigma_n\rangle^{\text{out}} \xrightarrow[x_i^0|_{\min}\to+\infty]{} \sum_{\pi}\prod_i \langle 0|\,\mathcal{O}_{A_i}(x_i)\,|q_{\pi(i)}\sigma_{\pi(i)}\rangle = \sum_{\pi}\prod_i z^{s}\,\psi_{A_i}^{\sigma_i}(p)\, e^{-iq_{\pi(i)}x_i}}$$

($\sigma_i = -s\dots+s$)

$\left(\text{and analogous for } {}^{\text{in}}\langle \omega_1\sigma_1\dots\omega_m\sigma_m|\,T\mathcal{O}\dots|0\rangle\right)$ $\quad$ Res F.T. $\langle\ \rangle \propto \psi_A^{\sigma_i}(p)\, z^{s}_{\dots\sigma_i\dots}$

The residues of F.T. $\langle\ \ \rangle$ give S-matrix elements contracted with various wavefunctions $\psi_A^{\sigma}(p)$. To get just $\mathcal{S}$-matrix we need to use the pseudounitary conditions (23 a-b) of L2

(40) $$\boxed{\sum_{A\bar{A}} C_{\bar{A}}^{*\,J'J_3'}\, \mathbb{P}^{\bar{A}A}\, C_A^{JJ_3} = \delta^{J}_{J'}\, \delta^{J_3}_{J_3'}\, \mathbb{P}^{(J)}}$$

parity in (j_-, j_+) — and in J-irreps. $(\mathbb{P}^{(J)2} = 1,\ \mathbb{P}^2 = 1)$

$\Longrightarrow$ reduce to scalar case by working with

(41) $$\boxed{\begin{aligned} &\vartheta^{s\sigma}(x) \equiv C_{\bar{A}}^{*\,s\sigma}\, \mathbb{P}_{\bar{A}B}\, D^{(j_-,j_+)}_{BC}(L_p^{-1})\, \mathbb{P}^{(s)}\, \vartheta_c^{(j_-,j_+)}(x) \\ &\langle 0|\vartheta^{s\sigma}(0)|p\,\sigma'\rangle_{s'} = \mathbb{P}^{(s)} C_{\bar{A}}^{*\,s\sigma}\, \mathbb{P}_{\bar{A}B}\, \underbrace{\cancel{D^{(j_-,j_+)}_{BC}(L_p^{-1})}\, \cancel{D^{(j_-,j_+)}(L_p)_{CD}}}_{\delta_{BD}}\, C_D^{s\sigma} \underset{40)}{=} \delta^{\sigma\sigma'} \end{aligned}}$$

repeated indices summed over

Moreover, $D^{(j_-,j_+)}(L_p^{-1})$ can be obtained directly from $D^{(j_+,j_-)}(L_p)^{\dagger}$

(42) $$\begin{cases} D^{(1/2,0)}(L_p) = e^{i\frac{\vec\sigma}{2}\vec\theta - \frac{\vec\sigma}{2}\vec\alpha} \\ D^{(0,1/2)}(L_p) = e^{i\frac{\vec\sigma}{2}\vec\theta + \frac{\vec\sigma}{2}\vec\alpha} \end{cases} \qquad \left(D^{(1/2,0)}(L_p)\right)^{\dagger} = e^{\left(-i\frac{\vec\sigma}{2}\vec\theta - \frac{\vec\sigma}{2}\vec\alpha\right)} = D^{(0,1/2)}(L_p^{-1}) \quad \text{i.e.} \begin{cases} (D^L)^{\dagger} = (D^R)^{-1} \\ (D^L)^{-1} = (D^R)^{\dagger} \end{cases}$$

(because $\mathbb{P}\vec{J}\mathbb{P} = \vec{J}$ and $\mathbb{P}\vec{K}\mathbb{P} = -\vec{K}$)

Example: 4-Vector A_μ and spin-1 particles

$$\langle 0|A_\mu(0)|\bar{p}\,\sigma\rangle_{s=1} = \varepsilon_\mu^\sigma(\bar{p}) \qquad \varepsilon_\mu^\sigma(p) = L_{p\,\mu}{}^{\nu}\, \varepsilon_\nu^\sigma(\bar{p})$$

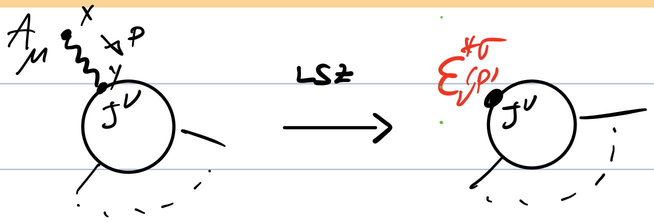
(43) $$-\varepsilon_\mu^{*\sigma'}(\bar{p})\, \eta^{\mu\nu} (L_p^{-1})_\nu{}^{\rho} A_\rho(0) \equiv A^{\sigma'}(0) \quad \Rightarrow$$

(44) $$\langle 0|A^{\sigma'}(0)|p\,\sigma\rangle_{s=1} = \underset{\mathbb{P}^{(s=1)}}{-}\varepsilon_\mu^{*\sigma'}(\bar{p})\, \eta^{\mu\nu} \underbrace{(L_p^{-1})_\nu{}^{\rho}\, \varepsilon_\rho^\sigma(p)}_{\varepsilon_\nu^\sigma(\bar{p})}\, Z^{s=1}(m^2) = -\underbrace{\varepsilon_\mu^{*\sigma'}(\bar{p})\, \varepsilon^{\sigma\mu}(\bar{p})}_{-\delta^{\sigma\sigma'}}\, Z^{(s=1)}(m^2) = +\delta^{\sigma\sigma'} Z^{(s=1)}(m^2)$$

ok

The net effect is cutting out the propagator, both denominator $\frac{|Z|^2}{p^2-m^2+i\varepsilon}$ and numerator, and replacing with wavefunc.

(45)



i.e. amputed$_\nu$ · $\varepsilon^{*\sigma}_{\nu}(p) = \langle 0|T J_\nu(p)\ldots|0\rangle \cdot \varepsilon^{*\nu}_{(p)}$
e.g.

LSZ operation; propagator

$$(46)\quad \left[-\varepsilon^{*\sigma'}_{\mu}(\bar p)\,(L_p^{-1})^{\mu}{}_{\nu}\,(-i(p^2-m^2))\right]\left(-\eta^{\nu\rho}+\frac{p^\nu p^\rho}{m^2}\right)\frac{i}{p^2-m^2+i\varepsilon} = -\varepsilon^{*\sigma'}_{\mu}(\bar p)\left(-\eta^{\mu\alpha}+\frac{\bar p^\mu \bar p^\alpha}{m^2}\right)(L_p)^{\rho}{}_{\alpha}$$

$\eta = \Lambda^T\eta\Lambda$
$\eta^{-1} = \Lambda^{-1}\eta^{-1}\Lambda^{T-1}$

$$= +\varepsilon^{*\sigma'\alpha}(\bar p)\,(L_p)^{\rho}{}_{\alpha} = +\varepsilon^{*\sigma'\rho}(p)$$

$\varepsilon(\bar p)\cdot\bar p = 0$

$$\boxed{\begin{array}{l}\text{LSZ}\cdot\text{Propagator} = \text{wavefunction}^* \\ \qquad\qquad\text{(for out-state)}\end{array}}$$

this is general via Källen-Lehmann

Källen-Lehmann; numerator prop. spin s

$$(47)\quad \langle 0|T\,\mathcal{O}^{(j_-,j_+)}_A(x)\,\mathcal{O}^{(j_-,j_+)\dagger}_{\bar B}(y)|0\rangle = \int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}\sum_s\int_0^\infty dM^2\,\frac{Z^{(s)}(M^2)\,i}{p^2-M^2+i\varepsilon}\left(D^{(j_-,j_+)}_{AC}(L_p)\,C^{s\sigma}_{C}\,C^{*s\sigma}_{\bar D}\,D^{(j_-,j_+)*}_{\bar B\bar D}(L_p)\right)$$

$\mathbb{1} = \sum_s\int dM^2\,|p\rangle_s\,{}_s\langle p|$ all states, not just 1-partic.

which we assume contains a pole $Z^{(s)}(M^2) = Z^{(s)}(m^2)\,\delta(M^2-m^2)+\ldots$

and multiplied by $\frac{(p^2-m^2)}{Z^{(s)}(m^2)}$ and by

$$\boxed{\begin{array}{l}\left(p^{(s)}C^{*s'\sigma'}_{\bar A}\,P_{\bar A B}\,D^{(j_-,j_+)}(L_p^{-1})\right)\cdot\left(\frac{p^2-m^2}{iZ^{(s)}(m^2)}\right)\ \text{F.T.}(59)\ \xrightarrow[p^2\to m^2]{}\ p^{(s)}\overbrace{C^{*s'\sigma'}_{\bar A}\,P_{\bar A A}\,C^{s\sigma}_{A}}^{\delta^{s'}_{s}\,\delta^{\sigma'}_{\sigma}}\,C^{*s\sigma}_{\bar D}\,D^{(j_-,j_+)*}_{\bar B\bar D}(L_p) \\ (48)\qquad = C^{*s'\sigma'}_{\bar D}\,D^{(j_-,j_+)*}_{\bar B\bar D}(L_p) = \psi^{s'\sigma'*}_{\bar B}(p)\end{array}}$$

this is for outstates created by $\mathcal{O}$
similar holds for instate w.r.t. $\mathcal{O}^\dagger$

In summary: LSZ cuts propagator & replace it with overlaps.

Example: Dirac Spinor $\Psi = \begin{pmatrix} \psi_L \\ \chi_R \end{pmatrix} = (1/2, 0) \oplus (0, 1/2)$

This is a reducible rep. $\psi \xrightarrow{\text{Lorentz}} \left(\begin{array}{c|c} D_L & 0 \\ \hline 0 & D_R \end{array}\right) \begin{pmatrix} \psi_L \\ \chi_R \end{pmatrix} \equiv D_D \psi$

(49) with $D_L^+ = D_R^{-1}$ & $D_R^+ = D_L^{-1} \Rightarrow \bar\Psi \equiv \psi^+ \gamma^0 \to \psi^+ \gamma^0 \gamma^0 \left(\begin{array}{c|c} D_R^{-1} & \\ \hline & D_L^{-1} \end{array}\right) \gamma^0 = \bar\Psi D_D^{-1}$
(and $\bar\Psi \psi$ is thus invariant).
Likewise, the $P = \gamma^0$ and the factor to multiply with is $u^+ \gamma^0 = \bar u$

(50) $\langle 0 | \Psi(0) | p\, \sigma \rangle = u^\sigma(p) = \left(\begin{array}{c|c} D_L(L_p) & \\ \hline & D_R(L_p) \end{array}\right) u^\sigma(\bar p) = D_D(L_p)\, u^\sigma(\bar p)$

(51) $\bar u^{\sigma'}(p) = \bar u^{\sigma'}(\bar p)\, D_D^{-1}(L_p)$

$\Downarrow$

(52) $\bar u^{\sigma'}(p)\, u^\sigma(p) = \bar u^{\sigma'}(\bar p)\, u^\sigma(\bar p) \propto \delta^{\sigma\sigma'}$

Field Redefinitions

The LSZ is a powerfull non-perturbative statement (never used a Lagrangian or fields being fundamental) that relies only on non-vanishing overlap $\langle 0 | \mathcal{O}(0) | 1\text{-particle} \rangle \neq 0$ (and $\langle 0 | \mathcal{O}(0) | 0 \rangle = 0$). Apart from this, we can choose $\mathcal{O}$ very different, (pion field $\pi(x)$ or $\bar q q$ pair of quark-antiquark) a freedom that it's often usefull in perturbative calculations too.

Exemple: $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 F(\phi)$ with $F(\phi) = 1 + c_1\phi + c_2\phi^2 + \ldots > 0$

search: $\phi' = \phi'(\phi)$ $\quad \partial_\mu \phi' = \frac{\partial \phi'}{\partial \phi}\partial_\mu \phi$ $\quad (\partial_\mu \phi')^2 = \left(\frac{\partial \phi'}{\partial \phi}\right)^2 (\partial_\mu \phi)^2$

(53) $$\boxed{\frac{\partial \phi'}{\partial \phi} = \sqrt{F(\phi)} = 1 + \frac{c_1}{2}\phi + \ldots} \qquad \phi'(\phi) = \int^\phi d\tilde{\phi}\,\sqrt{F(\tilde{\phi})} = \phi + \frac{c_1 \phi^2}{4} + \ldots$$

$$\Downarrow$$

$$M(12 \to 34) = 0$$

check with original Lagrangian $M(12 \to 34) \propto \Sigma_i\, p_i\, p_j \propto (s+t+u) = 0$ by symm. permut.

What if massive instead? $\to$ Exercise.

— Field Redefinitions vs. Equations of Motion & Redundant Operators —

Take an action expandable in some small parameter ϵ (couplings, $\frac{\partial}{\Lambda}, \ldots$) derivative exp.

(54) $$S[\phi] = S_0[\phi] + \epsilon\, S_1[\phi] + \epsilon^2 S_2[\phi] + \cdots$$

and that a certain contribution in S_m vanishes on lowest order e.o.m. $\frac{\delta S_0}{\delta \phi} = 0$

(55) $$S_m[\phi] = \int d^4x\, \frac{\delta S_0[\phi]}{\delta \phi} f(\phi(x), \partial\phi) + \hat{S}_m$$

classically $= 0$ on e.o.m.

This term is called redundant operator (to level m): it affects $O(\epsilon^{m+1})$

(56) $\phi \xrightarrow{\text{field redefines}} \phi - \varepsilon^m f(\overbrace{\phi, \partial_\mu \phi}^{\phi^2,\ \partial\phi\partial\phi,\ \dots})$ so that the action changes

(57) $S \longrightarrow S_0 + \varepsilon S_1 + \dots + \varepsilon^{m-1} S_{m-1} + \underbrace{\varepsilon^m \left[S_m - \frac{\delta S_0}{\delta \phi} f(\phi, \partial\phi) \right]}_{\hat{S}_m \text{: without the vanishing term on e.o.m.}} + O(\varepsilon^{m+1})$

(58) $S_{n<m} \to S'_{n<m}$, $S'_m \to \hat{S}_m$, $S_{n>m} \to S'_n$

Example: $\mathcal{L}_1 = \frac{1}{2}(\partial\phi)^2 - V(\phi) + \frac{1}{\Lambda^2}\phi^3 \Box\phi \underset{\text{Equiv.}}{\Rightarrow} \mathcal{L}_2 = \frac{1}{2}(\partial\phi)^2 - V(\phi) + \frac{\phi^3}{\Lambda^2}\underbrace{\cancel{(\Box\phi + V'(\phi))}}_{\text{redundant}} - \frac{\phi^3}{\Lambda^2} V'(\phi) + O\left(\frac{1}{\Lambda^4}\right)$

remove it by $\phi \to \phi + \frac{\phi^3}{\Lambda^2}$, since $\frac{\delta S_0}{\delta \phi} = -\Box\phi - V'\phi$.

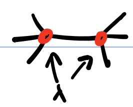
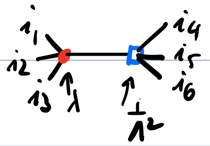
Effectively, just the same as using S_0-e.o.m. $\Box\phi = -V'$ plug-in interact

check: with $V(\phi) = \frac{\lambda}{4!}\phi^4$ $\quad V'(\phi) = \frac{\lambda}{3!}\phi^3$ in $\mathcal{L}_1$ & $\mathcal{L}_2$

(59) $\mathcal{L}_1$: $M_{12\to 34}\big|_{\text{tree}} = -i\lambda$ because $\phi^3\Box\phi$-vertex gives $\propto p_1^2 + p_2^2 + p_3^2 + p_4^2 = 0$

$\mathcal{L}_2$: $M_{12\to 34}\big|_{\text{tree}} = -i\lambda$ because $-\frac{\phi^3}{\Lambda^2}V'(\phi) = -\frac{\phi^6}{3!\Lambda^2}$ enters at 1-loop in $12 \to 34$

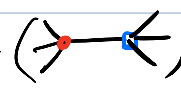
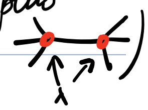
What about $M_{123\to 456}$ at $O(1/\Lambda^2)$?

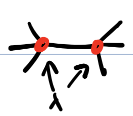
In $\mathcal{L}_1$:  +  ← only $\Box\phi$ internal contributes; if external $\Box \to p_i^2 \approx 0$ (that's why $3!$ from $\frac{\phi^3\Box\phi}{\Lambda^2}$)

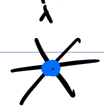
(60) one particular such diag: $\left(\frac{-i\lambda}{4!}\right) 4!\, 3! \left(\frac{-i}{\Lambda^2}\right)\cancel{(K_{i_4}+K_{i_5}+K_{i_6})^2} \frac{i}{\cancel{(K_{i_1}+K_{i_2}+K_{i_3})^2}} = \frac{-i\lambda \cdot 3!}{\Lambda^2}$

(one particular mom. partition)

There are $\frac{6\cdot 5\cdot 4}{3!} = \frac{6!}{3!\,3!}$ of these partitions of 3 momenta $\Rightarrow$ the contribut.

(61) to the amplitude from $\Sigma\Big($$\Big) = \frac{-i\lambda}{\Lambda^2}\frac{3!\,6!}{3!\,3!} = \frac{-i\lambda}{\Lambda^2}\frac{6!}{3!}$ (plus 

In $\mathcal{L}_2$:  still there, but now there is a ϕ^6-vertex

(62)  $= -i\frac{\lambda}{3!\Lambda^2}\,6!$ much easier!!